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UNSTEADY NONLINEAR CAPILLARY WAVE IN A LAYER OF VISCOUS LIQUID

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Dynamics of a viscous liquid layer is considered in nonlinear formulation with allowance for capillary forces. The problem of propagation of a wave, induced by the reduced pressure at the layer boundary, into the layer is solved .
The problem of perturbation propagation in a liquid layer was earlier considered in [1] in a linear formulation with allowance for capillary forces.

1. Statement of the problem. Let the thickness h of a viscous liquid layer contiguous at least on one side to a gas vary along the characteristic distance $l \gg h$ and let conditions $h^2 \ll \nu\tau$ and $h^2 v / (l\nu) \ll 1$ where ν is the kinematic viscosity, τ is the characteristic time of variation of h and v the velocity of liquid in the layer, be satisfied. Then the flow of liquid Q through a unit cross section of the layer differs only slightly from the steady flow in a flat slot [1].

$$\mathbf{Q} = -(h^3 k / 3\mu)\operatorname{grad} p, \quad p = p_0 - \sigma n \Delta h, \quad k = 1, {}^1/_4 \quad (1.1)$$
$$n = 1, {}^1/_2$$

where μ is the dynamic viscosity, σ is the coefficient of surface tension, and pressure p differs from the gas pressure p_0 by the capillar pressure. If one of the surfaces is bounded by a solid body and the other is free, the parameter k in (1.1) is unity, and $k = {}^1/_4$ when the tangential velocity is zero on both surfaces, which is possible in the presence of surface-active substances. Parameter $n = 1$ if the layer is bounded by a solid surface, and $n = {}^1/_2$ when the two boundaries are free.

Adding to (1.1) the equation of mass conservation

$$\operatorname{div} \mathbf{Q} + \partial h / \partial t = 0 \quad (1.2)$$

we obtain, as in [1], the equation for $h(x, t)$ (x is the coordinate along the layer). In the plane problem the equation is of the form

$$\varkappa \frac{\partial}{\partial x}\left(h^3 \frac{\partial^3 h}{\partial x^3}\right) = -\frac{\partial h}{\partial t}, \quad \varkappa = \frac{\sigma k n}{3\mu} \quad (1.3)$$

Solutions of a number of a number of problems related to this equation were investigated in linear approximation in [1]. It is interesting to solve the following nonlinear boundary value problem (e.g., in connection with problems of coalescence of bubbles or the dynamics of foam films).

Let the homogeneous in one direction layer of thickness h_∞

$$p \to p_0, \quad h \to h_\infty, \quad x \to \infty \quad (1.4)$$

be contiguous to a region of constant reduced pressure (the meniscus), where $p = p_0 - p_\sigma$ and, consequently,

$$\sigma n \partial^2 h / \partial x^2 \to p_\sigma, \quad x \to -\infty \quad (1.5)$$

In that region the layer thickness increases indefinitely

$$h \approx a_0 x^2 + a_1 x + \text{const} + \ldots, \quad x \to -\infty$$

The coefficient a_0 is simply expressed in terms of pressure drop p_σ. Specification of the second coefficient a_1 is primarily dependent on the fixing of the meniscus position along the x-axis. Selection of a_1 is here arbitrary, since in conformity with condition (1.4) the problem is considered with an accuracy to within the translation.

We have to determine the asymptotic form of function $h(x, t)$ when $t \to \infty$ which is independent of initial conditions at $t = 0$.

2. Self-similar solution. Let us consider the subsidiary problem which has a real physical meaning. Let instead of (1.5) the conditions

$$h = 0, \quad Q = 0, \quad x = 0 \tag{2.1}$$

be satisfied. These conditions correspond to a compression of the layer to zero thickness at point $x = 0$. Since the parameter of length dimension is absent, a self-similar solution is possible in conformity with the theory of dimensions [2]. It is of the form

$$h = h_\infty y(\zeta), \quad \zeta = x (4 \varkappa h_\infty{}^3 t)^{-1/4} \tag{2.2}$$

Equation

$$(y^3 y''')' = \zeta y'; \quad y \to 1, \ \zeta \to \infty \tag{2.3}$$

corresponds to Eq. (1.3) and condition (1.4).

Condition

$$y = 0, \quad y^3 y''' = 0, \quad \zeta = 0 \tag{2.4}$$

corresponds to condition (2.1) with allowance for (1.1).

The behavior of solution of the boundary value problem (2.3), (2.4) when $\zeta \to \infty$ is of considerable interest. Investigation of the linearized equation (2.3) at the limit $\zeta \to \infty$, shows that the asymptotics of y depends on two constants c_1 and c_2

$$y = 1 + c_1 \zeta^{-2/3} \exp\left(-{}^3/_8 \zeta^{4/3} - {}^{31}/_{72} \zeta^{-4/3} - {}^3/_4 \zeta^{-8/3} + \ldots\right) \times$$
$$\cos\left(c_2 + {}^3/_8 \sqrt{3} \zeta^{4/3} - {}^{31}/_{72} \sqrt{3} \zeta^{-4/3} + {}^3/_4 \sqrt{3} \zeta^{-8/3} + \ldots\right)$$

These two constants can be determined by satisfying two conditions (2.4).

The asymptotics of solution of Eq. (2.3) that satisfy (2.4) for $\zeta \to 0$ can be determined by the method of successive approximations

$$y = b\zeta + \frac{\zeta^2}{4b^2} (\ln \zeta + d) + \frac{\zeta^3}{144 b^5} \left(-5 \ln \zeta - 5d + \frac{59}{6}\right) + \ldots \tag{2.5}$$

The simplest way of finding the constants b and d which appear in the last equation is by taking into account the invariant transformation of Eq. (2.3)

$$y = \theta^{4/3} y_1, \quad \zeta = \theta \zeta_1 \tag{2.6}$$

The constants in (2.5) are transformed by formulas

$$b = \theta^{1/3} b_1, \quad d = d_1 - \ln \theta$$

For $b_1 = 1$ it is sufficient to use numerical calculations for solving Eq. (2.3), extend the asymptotics (2.5), and determine d_1, for which $y_1 \to y_\infty = \text{const}$ when $\zeta \to \infty$. The invariant transformation (2.6) with parameter $\theta = y_\infty^{-3/4}$ provides the solution of the input boundary value problem (2.3), (2.4). As the result we obtain

$$b = 0.628, \quad d = -6.70 \tag{2.7}$$

The curve $y(\zeta)$ is shown in Fig. 1.

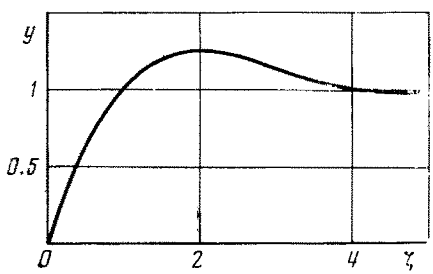

Fig. 1

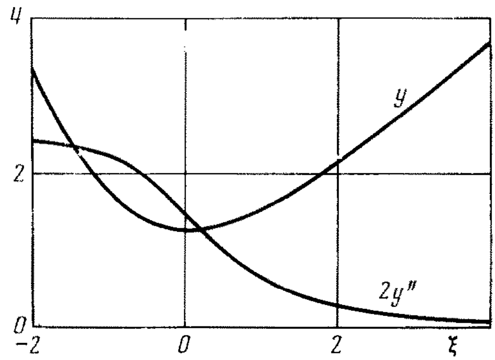

Fig. 2

3. The approximate solution. In the region where the layer becomes a meniscus the variation of thickness h follows a parabola we seek an approximate solution of Eq. (1.3) that satisfies condition (1.5) and condition

$$\partial^2 h / \partial x^2 \to 0, \quad x \to \infty \tag{3.1}$$

instead of condition (1.4). After that we carry out the joining.

We integrate both sides of (1.3) from $x = x_0$ where the thickness is minimum

$$\varkappa h^3 \frac{\partial^3 h}{\partial x^3} = Q_0 - \int_{x_0}^{x} \frac{\partial h}{\partial t} dx, \quad Q_0 = Q(x_0, t) < 0 \tag{3.2}$$

On the assumption (which is proved below) that

$$|Q_0| \gg \left| \int_{x_0}^{x} \frac{\partial h}{\partial t} dx \right| \tag{3.3}$$

problem (3.1), (3.2), (1.5) can be reduced to the following form:

$$\varkappa h^3\partial^3 h / \partial x^3 = -|Q_0(t)|; \quad \partial h / \partial x \to c(t), \quad x \to \infty$$
$$n\sigma\partial^2 h / \partial x^2 \to p_\sigma, \quad x \to -\infty \tag{3.4}$$

where functions $Q_0(t)$ and $c(t)$ are so far arbitrary. Problem (3.4) has a solution of the form

$$h = f(t)y(\xi), \quad \xi = (x - x_0) / l(t)$$
$$l = (n\sigma / p_\sigma)q_0 c, \quad f = (n\sigma / p_\sigma)q_0 c^2 \tag{3.5}$$

Constant q_0 and function $y(\xi)$ are determined by the solution of the boundary value problem

$$y^3 y''' = -1; \quad y'' \to q_0; \quad \xi \to -\infty; \quad y' \to 1, \quad \xi \to +\infty \tag{3.6}$$

When $\xi \to \infty$ the asymptotics $y(\xi)$ is of the form

$$y(\xi) = \xi - {}^1/_2 \ln \xi + {}^1/_4\xi^{-1}(\ln \xi + {}^{11}/_6) + {}^1/_{16}\xi^{-2}(\ln^2 \xi + {}^5/_3 \ln \xi + {}^5/_{36}) + \ldots \tag{3.7}$$

The numerical solution of problem (3.6) with allowance for (3.7) makes it possible to determine $y(\xi)$ (see Fig. 2) and, also, q_0 and the minimum value of $y(\xi)$

$$q_0 = 1.210, \quad y_{\min} = 1.259 \tag{3.8}$$

The minimum layer thickness $h_{\min}$ and the flow Q_0 are

$$h_{\min} = \frac{n\sigma}{p_\sigma} q_0 c^2 y_{\min}, \quad Q_0 = -\frac{n\sigma}{p_\sigma}\varkappa q_0 c^5 \tag{3.9}$$

Using formulas (3.5) and asymptotics (3.7) we find that condition (3.3) is satisfied in region

$$2\varkappa n\sigma q_0 c^5 \gg p_\sigma |dc / dt| (x - x_0)^2 \tag{3.10}$$

It will be seen from (3.7) or Fig. 2 and the similar asymptotic representation for $\xi \to -\infty$, that the scale of the obtained solution is $|\xi - \xi_0| \sim 1$ or $|x - x_0| \sim l$. Taking this into account, from (3.5) and (3.10) we obtain the following condition of the existence of the derived approximate solution:

$$2\varkappa p_\sigma c^3 \gg n\sigma q_0 |dc / dt| \tag{3.11}$$

4. Dynamics of a semi-infinite layer. The solutions derived in Sections 2 and 3 make it possible to construct the combined solution of the problem formulated in Section 1. We introduce the intermediate scale λ

$$\lambda^2 \mid dc \,/\, dt \mid = 2q_0 \varkappa c^5 N, \quad N = n\sigma \,/\, p_\sigma \tag{4.1}$$

in which Q_0 and the integral in (3.2) are of the same order. If $x - x_0 \gg \lambda$, the quantity Q_0 in (3.2) is negligibly small, and condition (2.1) is valid at point $x = x_0$. If furthermore

$$x - x_0 \ll L = (4\varkappa h_\infty^3 t)^{1/4}, \quad x_0 = 0 \tag{4.2}$$

where L is the scale of the self-similar solution (2.2), then the asymptotic representation (2.5) is valid. Joining the first terms of the two asymptotic representations of h by formulas (3.7), (3.5), (2.5), and (2.2) and allowing for (2.7) and (3.8), we obtain

$$c = b\chi^{1/4}, \quad \chi = h_\infty \,/\, (4\varkappa t) \tag{4.3}$$

As the result we have the composite solution that in region $x - x_0 \gg \lambda$ is determined by formula (2.2) and in region $x - x_0 \ll \lambda$ by formula (3.5). The parameters of solution (3.5) and (3.9) are as follows:

$$\begin{aligned} &l = q_0 b N \chi^{1/4}, \quad f = q_0 b^2 N \chi^{1/2} \\ &h_{\min} = y_{\min} q_0 b^2 N \chi^{1/2}, \quad Q_0 = -\varkappa q_0 b^5 N \chi^{5/4} \\ &N = n\sigma \,/\, p_\sigma, \quad \chi = h_\infty \,/\, (4\varkappa t) \end{aligned} \tag{4.4}$$

where l is the characteristic scale of the transition region between the nonlinear wave (2.2) and the meniscus $h_{\min}$ is the minimum thickness, and the constants q_0, b and $y_{\min}$ are determined by (2.7) and (3.8). The obtained solution is diagrammatically represented in Fig. 3 for the consecutive instants of time 1, 2, 3. Simultaneously with the penetration of the wave into the layer in conformity with the law $L \sim t^{1/4}$ the layer thickness in the continuously narrowing $(l \sim t^{-1/4})$ boundary region decreases proportionally to $t^{-1/2}$.

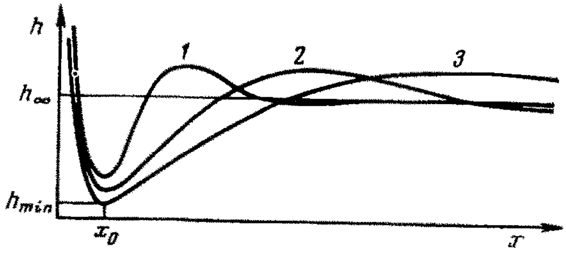

Fig. 3

The condition of validity of solution obtained in Section 2 consists of the existence of region (4.2). The inequality (4.2) is possible when $L \gg \lambda$. From (4.1)-(4.3) follows the condition

$$2\,(\varkappa h_\infty t)^{1/4} \gg \sqrt{n\sigma / p_\sigma}$$

It can be shown that condition (3.11) of validity of solution obtained in Sect. 3 is simultaneously satisfied.

5. The film of finite dimensions. Let a round film of radius r be contiguous to the meniscus along its circumference in which the pressure is reduced by p_σ as compared to the pressure of gas. At the initial instant $t = 0$ the film is of uniform thickness. It is obvious that at instants of time at which the scale L of the wave induced by the lower pressure at the border is considerably smaller than the film radius r the problem of the film dynamics reduces to the plane problem whose solution was obtained above. The condition $L \sim r$ yields the characteristic time +

$$\tau \approx r^4 / (4\varkappa h_\infty^3) \tag{5.1}$$

taken for the film to commence thinning at its center from the beginning of the process.

Formula (5.1) is the same, except for the unimportant numerical multiplier, as the similar formula for the time during which a perturbation in the layer propagates over distance r obtained in [1].

For $t \approx \tau$ the minimum thickness at the film borders is of order

$$h_{\min} \approx (n\sigma / p_\sigma) 0.6\, h_\infty^2 / r^2 \tag{5.2}$$

It can be expected that at instants of time $t \gg \tau$ the pressure in the film at distances of the order of r varies slightly, the wave degenerates, and the solution is close to

$$h = h_0(t)\,(1 - z^2 / r^2) \tag{5.3}$$

where z is the distance from the center of the film. A particular solution of Eqs. (1.1) and (1.2) close to (5.3) is possible if

$$64 h_0^4 \varkappa \gg r^4\, |\, dh_0 / dt\, | \tag{5.4}$$

From (1.1) and (1.2) follows

$$\varkappa h^3 \frac{\partial}{\partial z}\left(\frac{1}{z}\frac{\partial}{\partial z} z \frac{\partial h}{\partial z}\right) = -\frac{1}{z}\int_0^z z \frac{\partial h}{\partial t}\, dz \tag{5.5}$$

At the film border at $z \approx r$ the scale of thickness variation is $l \ll r$ and in the left-hand side of Eq. (5.5) the derivative

$$\varkappa h^3 \frac{\partial^3 h}{\partial x^3} = \frac{1}{r}\int_0^r z \frac{\partial h}{\partial t}\, dz, \qquad x = r - z \tag{5.6}$$

is the principal term. The boundary condition with allowance for (5.3) is of the form

$$\partial^2 h / \partial x^2 \to 0, \quad \partial h / \partial x \to 2h_0 / r, \quad x \to +\infty \tag{5.7}$$

The problem (5.6), (5.7), (1.5) of determination of the transition region between the film and the meniscus is equivalent to the boundary value problem (3.4). The point at which thickness h has its minimum is $z = r$ hence $x_0 = 0$. The integral in (5.6) is calculated with asymptotic accuracy with the use of (5.3). Taking into account formulas (3.9) we obtain

$$h_0 = \left(\frac{p_\sigma r^6}{2^9 q_0 n\sigma\varkappa t}\right)^{1/4}, \quad h_{\min} = \frac{q_0 n\sigma}{p_\sigma}\left(\frac{2h_0}{r}\right)^2 y_{\min} \tag{5.8}$$

where parameters q_0 and $y_{\min}$ are determined in (3.8), and by its order of magnitude formula (5.8) corresponds to (5.2). Condition (5.4) is satisfied when $p_\sigma r^2 \gg h_0\sigma$.

In the region of considerable h, which includes the meniscus, the free surface is determined by equations of equilibrium. In a particular case the free surface outside the film and the transition region of dimension $\sim l$ can be a sphere of radius R. In that case $p_\sigma \approx 2\sigma / R$. If furthermore the case is restricted to a liquid layer bordering on a solid surface $(\varkappa = \sigma / 3\mu,\ n = 1)$, then formulas (5.8) are closer to the similar formulas in [3] derived on intuitive considerations.

It is not possible to agree with the assertions in [4] about the errors of the authors of paper [3]. On the contrary it is paper [4] that is erroneous.

An attempt was made in [4] to formulate some iterative method of solving equations of thin layer dynamics. If the equation for h is presented in the form $Ah = \partial h / \partial t$, where A is a differential operator which depends only on coördinates, that method can be briefly written as $Ah_{k+1} = \partial h_k / \partial t,\ k = 0, 1, \ldots$ The zero approximation is defined by $h_0 = f(t)$. The authors confine themselves to the calculation of parameter h_1 and identify this with the approximate solution.

Such method of solution is erroneous for the following reasons.

First, that method stipulates unconditionally that function h which is being approximated must a fortiori differ only slightly from $h_0 = f(t)$, in particular, because only the first approximation is used. This means that the layer thickness must vary only slightly along the coordinates. However, the authors endeavor to analyze problems in which h varies substantially along the coordinates.

Second, the authors of [4] stress that the fundamental equation of their investigation, which can be briefly presented in the form $Ah_1 = \partial h_0 / \partial t \equiv \partial f / \partial t$, can be considered as obtained on the simplifying assumption that the radial flow $Q(t, r) = a(t) r$. But this is a wrong assumption. Section 5 of the present work clearly shows that $Q(t, r) = \frac{1}{2}(dh_0 / dt)(r^3 / 2l^2 - r)$, since the film is close to a lens (5.3) and not to a plane-parallel layer.

Third, the fundamental wave properties of solutions are completely lost in the method [4] with the result that thickness perturbations of the thin macroscopic layer propagate very slowly along it [1]. Investigations [4] contradict the exact results in [1] obtained in linear approximation. In particular, the thinning of the plane-parallel layer must begin at the borders. While according to [4] it begins right at the center of the layer. This erroneous result is due only to the incorrect method [4].

The region of validity of the hydrodynamic theory is bounded by film of macroscopic thickness. The possibility of hydrodynamic formulation is lost when $h \sim 10^{-6}$ cm. For films of thickness $h \lesssim 10^{-5}$ cm the van der Waals forces must be taken into account.

The solutions obtained above are valid for investigating the problem of film destruction. In the absence of electrostatic repulsion of surface layers when $h_{min} \sim 10^{-5}$ cm the liquid will rapidly flow out from the region of minimum thickness under the action of van der Waals forces, and the film is destroyed in that narrow boundary region similarly to [5].

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